Solving on Sobolev Spaces

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Theme for Today

Let's take a step back and review some general tools and techniques from functional analysis to **solve equations**:

$$F(x) = y, \quad F : X \to Y, \quad x \in X, \quad y \in Y.$$

If $X = Y$ is a vector space, we can equivalently consider **fixed point equations**:

$$G_y(x) = x, \quad G_y : X \to X, \quad x \in X,$$

where $G_y(x) = F(x) + x - y$.

Constructing Solutions

Here are four common analytic approaches to examining and solving equations.

	existence	uniqueness	nonlinear
Contraction	✓	✓	✓
Compactness	✓	✗	✓
Solution Bounds	✗	✓	✓
Projections	✓	✓	✗

We shall focus on the approaches applicable to non-linear equations.

Banach Spaces

To find solutions, it is necessary to search within an appropriate space. To enable most approaches, an underlying complete metric space is required.

Definition

1. A metric space (X, d) is **complete** if every Cauchy sequence converges to an element in X.
2. A **Banach space** $(X, \|\cdot\|)$ is a normed vector space which is complete under the metric $d(x, y) = \|x - y\|$.

Banach Spaces

Example (Euclidean space)

$\mathbb{R}^n$ is a Banach space under the ℓ^p norm $\|x\|_{\ell^p}^p = \sum_{i=1}^n |x_i|^p$ or max norm $\|x\|_{\ell^\infty} = \max |x_i|$.

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Example (Continuous functions)

For a compact (Hausdorff) space Ω, the space of continuous functions $f : \Omega \to X$ where X is a Banach space, denoted $C(\Omega; X)$ (or just $C(\Omega)$ for $X = \mathbb{R}$) is a Banach space under the uniform norm

$$\|f\|_\infty = \sup_{x \in \Omega} \|f(x)\|.$$

Banach Spaces

Example (Dual spaces)

For any Banach space X, the space X^* of linear functionals $\varphi : X \to \mathbb{R}$ is a Banach space under the operator norm

$$\|\varphi\|_{X^*} = \sup_{\|x\| \leq 1} |\varphi(x)|.$$

Many useful spaces (e.g. space of measures) can be embedded into an appropriate dual space, often using integrals.

Banach Spaces

Example (L^p & Bochner spaces)

Let $\Omega \subseteq \mathbb{R}^n$ and X be a Banach space. For $1 \leq p \leq \infty$, the space $L^p(\Omega; X)$ is the quotient space (under equality a.e.) of measurable functions $u : \Omega \to X$ such that $\|u\|_{L^p(\Omega;X)} < \infty$ where

$$\|u\|_{L^p(\Omega;X)} = \begin{cases} \left(\int_\Omega \|u(t)\|^p dt \right)^{1/p} & \text{if } p < \infty \\ \inf\{M \geq 0 : \|u(t)\| \leq M \text{ a.e.}\} & \text{if } p = \infty \end{cases}$$

We will not be defining measures here, but note all integrals are defined under Lebesgue measure.

Banach Spaces

Example (Hölder spaces)

For $\alpha \in [0, 1]$ and integer k, the space of k-times differentiable functions which are finite under the α-Hölder norm

$$\|f\|_{C^{k,\alpha}(\Omega)} = \sum_{|\beta| \leq k} \|D^\beta f\|_\infty + \sum_{|\beta|=k} \sup_{x,y \in \Omega,\, x \neq y} \frac{\|D^\beta f(x) - D^\beta f(y)\|}{\|x - y\|^\alpha},$$

is a Banach space $C^{k,\alpha}(\Omega)$ under the same norm.

Sobolev Spaces

L^p spaces in the context of derivatives leads us **Sobolev spaces**. They are some of the most useful function spaces due to their powerful embedding theorems (more on this later).

For $\Omega \subseteq \mathbb{R}^n$ an open domain, integer k, and $p \geq 1$, the **Sobolev norm** for a smooth function u is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}.$$

Sobolev Spaces

Definition (Sobolev spaces)

- The Sobolev space $W^{k,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ under the $W^{k,p}$ norm.
- The "zero-boundary" Sobolev space $W_0^{k,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ under the $W^{k,p}$ norm.

Under this definition, both spaces are Banach spaces by construction.

Sobolev Spaces

An equivalent construction is by the concept of **weak derivatives**.

If u_m is Cauchy in $L^p(\Omega)$, then for any $\phi \in C_c^\infty(\Omega)$,

$$\left| \int_\Omega (u_m - u_n)\phi dx \right| \leq \|u_m - u_n\|_{L^p(\Omega)} \|\phi\|_{L^q(\Omega)} \to 0.$$

Therefore, **integration by parts** carries over to $u \in W^{k,p}(\Omega)$: for $|\alpha| \leq k$, there is $D^\alpha u \in L^p(\Omega)$ (called the α-**weak derivative**) such that

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi dx, \text{ for all } \phi \in C_c^\infty(\Omega).$$

Sobolev Spaces

Any function $u \in W^{k,p}(\Omega)$ is k-times weakly differentiable. On the other hand, due to **mollification**, we have

Theorem (Meyers-Serrin Theorem)

Let u be k-times weakly differentiable with all weak derivatives in L^p, that is,

$$\{D^\alpha u,\ |\alpha| \leq k\} \subset L^p(\Omega).$$

Then there exists a sequence of functions $u_m \in C^\infty(\Omega)$ with $\|u_m\|_{W^{k,p}(\Omega)} < \infty$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.

Sobolev Spaces

This connects the "approximation" definition with the previous one.

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- The special case $W^{k,2}$ is a Hilbert space, often denoted H^k.
- Sobolev spaces can also be defined on **Riemannian manifolds** (including the torus) in a similar manner, using appropriate integrals.
- For the torus $\mathbb{T}^n$, the space $H^k(\mathbb{T}^n)$ is most easily constructed using Fourier transforms via the inner product

$$\langle u, v \rangle_{H^k(\mathbb{T}^n)} = (2\pi)^k \sum_{x \in \mathbb{Z}^n} \hat{u}(x)\bar{\hat{v}}(x)(1 + \|x\|^2)^k.$$

Contraction

Theorem (Banach Fixed Point Theorem)

Let (X, d) be a complete metric space and $\Omega \subseteq X$ a non-empty closed subset. Suppose that $F : \Omega \to \Omega$ satisfies for some $0 < c < 1$,

$$d(F(x), F(y)) \leq cd(x, y), \qquad \text{for all } x, y \in \Omega.$$

Then there exists a **unique** $x^* \in \Omega$ solving the fixed point equation $x^* = F(x^*)$.

Any map F satisfying the conditions of the theorem is called a *contraction mapping*.

Contraction in ODEs

Corollary (Picard-Lindelof Theorem)

Let X be Banach, and $f(t,x) : [0,T] \times X \to X$ be L-Lipschitz continuous in x. There exists a time $0 < \epsilon < T$ such that there is a unique solution $x \in C([0,\epsilon]; X)$ to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \qquad 0 \leq t \leq \epsilon.$$

The Lipschitz criterion can be weakened to hold only in some bounded region, at the expense of smaller ϵ.

Contraction in ODEs

Proof.

Define $F : \mathcal{C}([0, \epsilon]; X) \to \mathcal{C}([0, \epsilon]; X)$, for $\epsilon > 0$ yet to be determined, by

$$F(x)(t) = x_0 + \int_0^t f(s, x(s))ds.$$

It will suffice to show F is a contraction. Note that

$$\begin{aligned} \|F(x)(t) - F(y)(t)\| &\leq \int_0^t \|f(s, x(s)) - f(s, y(s))\|ds \\ &\leq tL\|x - y\|_\infty. \end{aligned}$$

Therefore, $\|F(x) - F(y)\|_\infty \leq \epsilon L\|x - y\|$. Choosing $\epsilon < 1/L$, F is a contraction. $\square$

Compactness

In many cases, contraction is too much to ask for.

The most common and powerful approach for constructing solutions is by a **compactness argument**, which is used to show convergence of a chosen sequence of **approximations**.

The following definition is fundamental.

Definition

A subset S of a metric space (X, d) is (sequentially) *precompact* if every sequence in S has a convergent subsequence.

Compactness

Lemma

Let $S = \{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space (X, d). If S is precompact, and any limit point of $\{x_n\}_{n=1}^{\infty}$ must be x^*, then $x_n \to x^*$.

Proof.

If the sequence did not converge to x^*, then there would exist a subsequence of S that is at least distance ϵ away from x^*. But this subsequence would have a further subsequence converging to x^*, which is a contradiction. $\square$

Compactness

To prove that a sequence of approximate solutions f_k converges to a solution it suffices to show that

1. $\{f_k\}_{k=1}^{\infty}$ is precompact; and
2. any limit point of $\{f_k\}_{k=1}^{\infty}$ is a solution

This is the **compactness argument**.

Examples of Compactness Theorems

Theorem (Bolzano-Weierstrass)

Any bounded subset of $\mathbb{R}^d$ is precompact.

Theorem (Prokhorov)

A collection of measures M on X is precompact (under convergence in distribution) if and only if M is **tight**, i.e. for every $\epsilon > 0$, there is a compact set K_ϵ such that $\mu(X - K_\epsilon) < \epsilon$ for all $\mu \in M$.

Compactness for $C(\Omega)$ (mild solutions)

Theorem (Arzela-Ascoli)

A collection of continuous functions $\mathcal{F} \subset C(\Omega)$ is precompact if it is **uniformly bounded**:

$$\sup_{f \in \mathcal{F}} \|f\|_\infty < +\infty;$$

and **equicontinuous**: for every $\epsilon > 0$, there is $\delta > 0$ such that

$$\sup_{f \in \mathcal{F}} |f(x) - f(y)| < \epsilon, \quad \text{whenever } |x - y| < \delta.$$

Compactness in ODEs

Corollary (Peano Existence (global))

If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and continuous, then for any $T > 0$ and $x_0 \in \mathbb{R}^n$, there exists a solution to $x(t) = x_0 + \int_0^t f(s, x(s))ds$ in $\mathcal{C}([0, T]; \mathbb{R}^n)$.

Proof (Outline).

For each step size $h > 0$, let $x_h(t)$ denote the linear interpolation of the Euler scheme for the ODE. The family $\{x_{1/k}(t)\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous, and any limit point must be a solution by Dominated Convergence Theorem. $\square$

Sobolev Embeddings

The conditions of the Arzela-Ascoli theorem can often be too difficult to use, as equicontinuity may be too challenging to establish directly.

To perform the compactness argument with Sobolev spaces will require an understanding of **Sobolev embeddings**.

The corresponding Sobolev inequalities will also be extremely valuable for obtaining *a priori* estimates.

Isoperimetry

Let E be a bounded set in $\mathbb{R}^n$. The $(n-1)$-dimensional surface of E is denoted by ∂E.

The **isoperimetry inequality** states

$$\frac{m(\partial E)^{\frac{1}{n-1}}}{m(E)^{\frac{1}{n}}} \geq \frac{m(\partial B_r)^{\frac{1}{n-1}}}{m(B_r)^{\frac{1}{n}}} = \frac{m(\partial B_1)^{\frac{1}{n-1}}}{m(B_1)^{\frac{1}{n}}},$$

where B_r is a ball of radius r. In other words, **spheres have minimal surface area relative to volume**. In particular,

$$m(\partial E) \geq C_n m(E)^{1-\frac{1}{n}}.$$

Sobolev Embedding

- The **layer cake representation** (proven using Fubini's theorem) is

$$\int_{\mathbb{R}^n} |u(x)|^p dx = p \int_0^\infty s^{p-1} m(|u| > s) ds$$

- The **coarea formula** is

$$\int_{\mathbb{R}^n} \|\nabla u(x)\| dx = \int_0^\infty m(|u| = s) ds$$

The Sobolev inequalities are isoperimetry applied to level sets of functions

Sobolev Embedding

Theorem (Sobolev Inequality)

There exists a constant $C_n > 0$ such that

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_n \|u\|_{W^{1,1}(\mathbb{R}^n)}.$$

Consequently, for integers $0 \leq \ell < k$ and $\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{\ell}{n}$,

$$\|u\|_{W^{\ell,q}(\mathbb{R}^n)} \leq C_{p,k,\ell} \|u\|_{W^{k,p}(\mathbb{R}^n)}.$$

Therefore,

$$W^{k,p}(\mathbb{R}^n) \subset W^{\ell,q}(\mathbb{R}^n).$$

Compactness for $W^{k,p}$

Corollary (Rellich-Kondrachov Compactness)

For integers $0 \leq \ell < k$ and $\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{\ell}{n}$, if Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, any bounded subset of $W^{k,p}(\Omega)$ is precompact in $W^{\ell,q}(\Omega)$.

Trades

- Derivatives $\rightarrow$ higher-order integrability
- Derivatives $+$ integrability $\rightarrow$ compactness

Trading for Regularity

Consider a smooth function $u : \mathbb{R} \to \mathbb{R}$.

The fundamental theorem of calculus and Hölder's inequality implies that

$$\begin{aligned} |u(y) - u(x)| &\leq \int_x^y |u'(t)| dt \\ &\leq \left(\int_x^y |u'(t)|^p dt \right)^{1/p} \left(\int_x^y 1 dt \right)^{1-1/p} \end{aligned}$$

Trading for Regularity

Consider a smooth function $u : \mathbb{R} \to \mathbb{R}$.

The fundamental theorem of calculus and Hölder's inequality implies that

$$|u(y) - u(x)| \leq \|u'\|_{L^p(\mathbb{R})} |x - y|^{1-1/p}$$

Trading for Regularity

Consider a smooth function $u : \mathbb{R} \to \mathbb{R}$.

The fundamental theorem of calculus and Hölder's inequality implies that

$$\sup_{x \neq y} \frac{|u(y) - u(x)|}{|x - y|^{1-1/p}} \leq \|u'\|_{L^p(\mathbb{R})}.$$

Therefore, in one dimension, $W^{1,p}$ for $p > 1$ implies uniform (Hölder) continuity! The same idea holds in $\mathbb{R}^n$ too when $p > n$.

Compactness of $W^{k,p} \hookrightarrow C^{k-[n/p]-1,\gamma}$

Theorem (Morrey Inequality)

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Assume $k - [n/p] - 1 \geq 0$. Then there exists $0 < \gamma < 1$ such that

$$\|u\|_{C^{k-[n/p]-1,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Consequently, there also exists $0 < \beta < 1$ such that any bounded subset of $W^{k,p}(\Omega)$ is precompact in $C^{k-[n/p]-1,\beta}(\bar{\Omega})$.

Compactness without Arzela-Ascoli!

Weak Compactness

The final solution approach we shall discuss is more general, and arguably even more powerful.

Thus far, we have discussed compactness theorems under **strong** convergence. **Weak** compactness (and consequently, a weak solution) is far easier to show.

Definition (Weak Convergence)

Let X be a Banach space. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is **weakly convergent** to x, denoted $x_n \rightharpoonup x$, if for every $f \in X^*$, $f(x_n) \to f(x)$.

Weak Compactness

Theorem (Eberlein-Šmulian Theorem)

Any bounded subset E of a reflexive Banach space X ($X = X^{**}$) is precompact in the weak topology, that is, any sequence in E has a subsequence which converges weakly.

For $1 < p < \infty$, L^p spaces and Sobolev spaces are reflexive. By Riesz representation, $u_n \rightharpoonup u$ implies

$$\int_\Omega u_n(x)\varphi(x)dx \to \int_\Omega u(x)\varphi(x)dx,$$

for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$. This leads to **weak solutions**.

Weak Compactness

To upgrade to strong convergence (often for the purposes of proving convergence of norms), we can take advantage of the other compactness theorems.

Lemma

If $u_n \rightharpoonup u$ and $\{u_n\}_{n=1}^{\infty}$ is precompact in X, then $u_n \to u$ in $\bar{X}$.