Monotone Methods in Second-Order Semilinear Dirichlet BVPs

Liam Hodgkinson

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Background on Linear Elliptic PDEs

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where for some $\lambda > 0$,

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also assuming $a^{ij}, b^i, c \in C^{\alpha}(\bar{\Omega})$ and $c \geq 0$.

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Theorem (Gilbarg & Trudinger, Theorem 6.14)

For any $f \in C^{\alpha}(\bar{\Omega})$ and $\phi \in C^{2,\alpha}(\bar{\Omega})$, the Dirichlet problem

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Proof follows from the Schauder estimates, in particular, there is a $C > 0$ with

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left(\|u\|_{\infty} + \|f\|_{C^{\alpha}(\bar{\Omega})} + \|\phi\|_{C^{2,\alpha}(\partial\Omega)} \right).$$

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Let $K_{\phi} : C^{\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ be the solution map from f to u for fixed ϕ.

The Semilinear Problem

What about the semilinear extension

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Goal: extend work on linear elliptic PDEs to the semilinear case.

The Hammerstein Equation

Suppose $f \in C^{\alpha}(\bar{\Omega} \times \mathbb{R})$ and define the Nemytskii operator

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A Similar Problem (Hammerstein Integral Equation)

$$\text{Solve } u(x) = \int_{\Omega} k(x, y) f(y, u(y)) d\mu(y).$$

The Maximum Principle

Lemma (Monotonicity of K_ϕ)

K_ϕ is a monotone operator, that is, for any $f, g \in C^\alpha(\bar{\Omega})$,

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$$\begin{aligned} Lv = f - g &\leq 0 \quad \text{in } \Omega \\ v = \phi - \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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- Weak maximum principle for L implies that $v \leq 0$ in Ω.

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- Thus $K_\phi\{f_n\}_{n=1}^{\infty} \subset C^{2,\alpha}\left(\bar{\Omega}\right)$ has a convergent subsequence.

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Suppose $f : \mathbb{R} \to \mathbb{R}$ is monotone and continuous. For any $x_0 \in \mathbb{R}$, the sequence $\{f^{(k)}(x_0)\}_{k=0}^{\infty}$ converges to a fixed point of f if and only if it is bounded.

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To generalise to Banach spaces, we need a partial ordering.

For vector spaces, it suffices to define the set of positive elements.

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Definition

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For $x, y \in X$, we say

- $x \geq y$ if $x - y \in X_+$,
- $x > y$ if $x - y \in X_+ \setminus \{0\}$,
- $[x, y] := \{z \in X : x \leq z \leq y\}$.

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Remarks

- $x \not\geq y$ does not imply $x < y$.
- $x > y$, $z > 0$ does not imply $\exists \delta > 0$ such that $x - \delta z > y$.
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- If X_+ has nonempty interior $\mathcal{X}_+$, we write $x \gg y$ if $x - y \in \mathcal{X}_+$.
- If there is a norm $\|\cdot\|$ of X such that

$$0 \leq x \leq y \text{ implies } \|x\| \leq \|y\|,$$

then X_+ is *normal*.

Ordered Banach Spaces

Example

For $\Omega \subset \mathbb{R}^n$ compact, $C(\Omega)$ is a Banach space over $\mathbb{R}$ with the normal reproducing cone

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However, it is not normal and has empty interior for $p < +\infty$.

Method of Subsolutions and Supersolutions

Lemma

Let X be a real Banach space with normal cone X_+, subspace $E \subset X$, and $f : E \to E$ a compact operator satisfying

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If $x_0, \bar{x} \in E$ satisfy

1. $x_0 \leq \bar{x}$
2. $x_0 \leq f(x_0)$ (subsolution)
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then f has a maximal and minimal fixed point in $[x_0, \bar{x}] \cap E$ given by the limits of the sequences of iterates $\{f^{(k)}(\bar{x})\}_{k=0}^{\infty}$ and $\{f^{(k)}(x_0)\}_{k=0}^{\infty}$ respectively.

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- Since the sequence is monotone, the whole sequence must also converge.
- By continuity, the limit of $\{f^{(k)}(x_0)\}_{k=0}^{\infty}$ is a fixed point of f.
- Any fixed point in $[x_0, \bar{x}] \cap E$ is both a lower bound of $\{f^{(k)}(\bar{x})\}_{k=0}^{\infty}$ and an upper bound of $\{f^{(k)}(x_0)\}_{k=0}^{\infty}$.

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Theorem (Amann, 1971)

Suppose $f \in C^{1,\alpha}(\bar{\Omega} \times \mathbb{R})$ has f_u bounded from below. The semilinear elliptic Dirichlet BVP

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Proof.

- Assume F is monotone increasing: otherwise take $F(u) + \omega u$.
- Apply lemma for $u = (K_\phi \circ F)(u)$.

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A Simple Example

Proposition

The semilinear boundary value problem (in $n \geq 2$ dimensions)

$$\begin{aligned} -\Delta u &= e^u \qquad \text{in } B_1 \\ u &= 0 \qquad \text{on } \partial B_1. \end{aligned}$$

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- A subsolution is given by $u \equiv 0$.
- Consider the function $u = 1 - \sum_{i=1}^n x_i^2 \geq 0$.
  - $-\Delta u = 2n \geq e^u$ in B_1.
  - Thus, u is a supersolution of the BVP.

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  - $-\Delta u = 2n \geq e^u$ in B_1.
  - Thus, u is a supersolution of the BVP.
- Apply theorem with $u_0 = 0$ and $\bar{u} = 1 - \sum_{i=1}^n x_i^2$.



The Parabolic Case

Theorem (Sattinger, 1972)

Let $\Gamma_T = \Omega \times (0, T)$ for $T > 0$. Suppose $f \in C^1(\Omega \times \mathbb{R})$ has partial derivative f_u bounded from below. The Dirichlet BVP

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu + f(x, u) && \text{in } \Gamma_T \\ u &= \phi && \text{on } \partial\Gamma_T \end{aligned}$$

has a unique solution $u \in C^{2,\alpha}(\Gamma_T)$ if it possesses a subsolution u_0 and a supersolution $\bar{u}$ with $u_0 \leq \bar{u}$.

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We will only apply for the $\phi \equiv 0$ case.

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- By SMP, there is $\alpha > 0$ such that $\alpha e \leq K_0 g$ in $\bar{\Omega}$.



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- Count solutions by subdividing the cone into intervals where either concave or convex (Amann, 1972).