PROOF OF THE BERRY-ESSEEN THEOREM

Definition. A Gibbs process is a point process that is absolutely continuous with respect to a Poisson process.

Let $\mathcal{A}f(x) = f'(x) - xf(x)$ denote the Stein operator for a standard normal random variable ($Z \sim \mathcal{N}(0,1)$). The main objective is to bound $|\mathbb{E}\mathcal{A}f(W)|$ where $W = n^{-1/2}\sum_{i=1}^{n} X_i$. There is

$$\mathbb{E}\mathcal{A}f(W) = \mathbb{E}f'(W) - \mathbb{E}[Wf(W)].$$

Begin with the second term: first observe that since $\mathbb{E}X_i = 0$ and X_i, W_i are independent,

$$\mathbb{E}[X_i f(W_i)] = 0.$$

Therefore,

$$\mathbb{E}[Wf(W)] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[X_i f(W) - X_i f(W_i)]$$

Due to Taylor's Theorem (slide change),

$$\begin{aligned}\mathbb{E}[Wf(W)] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2 f'(W_i) + n^{-1/2}X_i^3 f''(\xi_i)\right]\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[f'(W_i) + n^{-1/2}X_i^3 f''(\xi_i)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[f'(W) - Wf(W)] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[f'(W) - f'(W_i) - n^{-1/2}X_i^3 f''(\xi_i)\right]\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[n^{-1/2}X_i f''(\eta_i) - n^{-1/2}X_i^3 f''(\xi_i)\right]\end{aligned}$$

By Holder's inequality, $\mathbb{E}|X_i| \leq \left(\mathbb{E}X_i^2\right)^{1/2} = 1 \leq \mathbb{E}\left|X_i^3\right|$. Therefore,

$$|\mathbb{E}\mathcal{A}f(W)| \leq \frac{\|f''\|_\infty}{\sqrt{n}}\left(\mathbb{E}|X_i| + \mathbb{E}\left|X_i^3\right|\right) \leq \frac{2\mathbb{E}\left|X_i^3\right|\|f''\|_\infty}{\sqrt{n}}.$$

The theorem follows from Stein's lemma:

$$|\mathbb{E}f(W) - \mathbb{E}f(Z)| \leq \frac{4\mathbb{E}\left|X_i^3\right|\|f'\|_\infty}{\sqrt{n}}$$