AN INFORMAL INTRODUCTION TO ROUGH PATHS

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In a previous discussion, I covered the construction of the general stochastic integral according to the Bichteler-Dellacherie theorem via semimartingales. While this theory is quite elegant in its current state, it is generally incompatible with the more usual analysis techniques often seen in other forms of calculus. The differences between the Ito and Stratonovich stochastic calculus formulations are particularly distressing in this regard. Furthermore, there is no proper understanding of how SDEs might be treated *pathwise*. But the biggest issues arise when one wishes to study notions of regularity for solutions to stochastic differential equations, and these problems become insurmountable when we try to extend stochastic calculus to studying stochastic partial differential equations (SPDEs), yielding a disjointed and unnatural theory as a result.

To show why the traditional theory of SDEs is insufficient, we will compare it with well-established theory of ordinary differential equations (ODEs). Before that, I will state the following definition, which will prove to be critical to the discussion. For simplicity, I limit myself to the one-dimensional case.

Definition. For $\alpha \in (0,1]$, a function f is said to be *α-Holder continuous* if there is some constant $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|^\alpha$ for all x, y. A function which is 1-Holder is said to be *Lipschitz continuous*. For $\alpha \in (0,1)$, we let C^α denote the class of α-Holder continuous functions, and assign to this space the α-Holder norm $\|f\|_\alpha$, defined as

$$\|f\|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Fact. *(Interesting): Only constants are α-Holder for $\alpha > 1$.*

Holder continuity is a critically important form of regularity of functions, implying not only uniform continuity, but a kind of rate of convergence on this as well. Consider an ordinary differential equation for a function y which is *driven* by a function g. Such an ODE is of the form

$$dy = f(y)\,dg \qquad \text{(or equivalently, } y'(x) = f(y)\,g'(x)\text{),}$$

which has the obvious comparison to $dY = f(Y)\,dX$ in stochastic calculus. Suppose that we would like to show existence/uniqueness of a solution to this equation as an initial value problem on $[0, T]$. By the Picard-Lindelof theorem, if f is Lipschitz continuous, then a solution to this ODE exists locally (for some $[0, \epsilon]$, $\epsilon < T$). Additionally, suppose that g_ϵ is some approximation to g of order ϵ (so $g_\epsilon \to g$ as ϵ goes to zero) and we were only able to solve $dy_\epsilon = f(y_\epsilon)\,dg_\epsilon$ instead. Again, solutions exist locally. But also, by Gronwall's inequality, there is a constant M such that

$$\|y - y_\epsilon\|_\infty \le M \|f\|_\infty \left\|g' - g'_\epsilon\right\|_\infty .$$

Thus, if g_ϵ is close to g, it follows that y_ϵ is close to y. This form of continuity of the solution map is essential in both numerical and analytical contexts — in particular, it is important for demonstrating algorithmic efficiency if one wishes to actually solve these equations in practice. The standard theory of stochastic calculus does not have a result of this form. The closest thing is the Wong-Zakai theorem, but this only applies in the Stratonovich formulation.

Questions of regularity also apply to stochastic differential equations, but are difficult to answer in the theory. A founding motivation for the theory of rough paths is a well-known result by Kolmogorov.

Theorem (Kolmogorov Continuity Theorem)**.** *Let X be a stochastic process such that*

$$\mathbb{E}\left[|X_t - X_s|^\alpha\right] \le |t - s|^{1+\beta}$$

for some $\alpha, \beta > 0$ for every $s, t > 0$. Then X is almost surely γ-Holder continuous for every $0 < \gamma < \frac{\beta}{\alpha}$.

In particular, we find that sample paths of Brownian motion are almost surely $\left(\frac{1}{2} - \epsilon\right)$-Holder continuous for small $\epsilon > 0$. So to construct some pathwise notion of stochastic integration (at least, in one dimension, for now), we can consider the problem of integrating an α-Holder continuous function with respect to a β-Holder continuous function.

Theorem. *For any two functions X, Y, the Young integral constructed by*

$$\int Y dX = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} Y_s X_{s,t}$$

where $X_{s,t} = X_t - X_s$ converges if X and Y are α- and β-Holder continuous and $\alpha + \beta > 1$.

Proof. Young's inequality gives $\left|\int_s^t Y dX - Y_s (X_t - X_s)\right| \le C |t - s|^{\alpha+\beta}$ where C is independent of s, t. □

If Y is sufficiently regular (e.g. absolutely continuous), then the integral with respect to Brownian motion exists. This is the Paley-Wiener integral. However, we cannot take the integral $\int W_t dW_t$, or indeed, even $\int f(W_t)\, dW_t$. The reason, as Terry Lyons — the founder of rough path theory — discovered, is because the limit for the Young integral relies on the assumption that for s, r very close, $F(X_r) \approx F(X_s)$. This can be thought of as a *zeroth*-order approximation to $F(X_r)$. If we take a *first*-order approximation instead, we would get

$$F(X_r) \approx F(X_s) + F'(X_s)(X_r - X_s)$$

and our integral becomes

$$\begin{aligned}\int F(X)\, dX &= \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} [F(X_s) X_t + F'(X_s)(X_t - X_s) X_t - F(X_s) X_s] \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left[F(X_s)(X_t - X_s) + F'(X_s) X_{s,t} X_t\right]\end{aligned}$$

so if we already knew $\mathbb{X}_{s,t} = \int_s^t X_{s,t} dX_t$, then we would expect that

$$\int F(X)\, dX = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left[F(X_s)(X_t - X_s) + F'(X_s)\mathbb{X}_{s,t}\right]$$

would be the 'better' integral, and indeed, this integral converges under weaker conditions.

Definition. For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, an α-Holder rough path is a pair $(X, \mathbb{X})$ where $X : [0, T] \to \mathbb{R}$ and $\mathbb{X} : [0, T]^2 \to \mathbb{R}$ satisfying the following conditions:

(1) X is α-Holder continuous
(2) $\mathbb{X}$ is 2α-Holder continuous in the sense that $\left|\mathbb{X}_{s,t}\right| \le C\,|t-s|^{2\alpha}$ for some C.
(3) For every triple of times (s, t, u), $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s)(X_t - X_u)$ (Chen's relations).

This last condition is highly algebraic in nature, and comes from the theory of Lie groups. It can be verified that any traditional choice of iterated integral for $\mathbb{X}_{s,t}$ will formally satisfy this condition. Without giving too much away, I'd like to point out that the restriction to $\alpha > \frac{1}{3}$ *is* significant; an analogous theory holds for smaller α, requiring higher-order 'approximations' and iterated integrals. I won't be considering these cases, however, as I'm primarily interested in looking at Brownian motion as a rough path, which is certainly α-Holder for $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

A natural question at this stage is whether any α-Holder function can be lifted to a rough path (although this process may not be unique). The answer to this is affirmative.

Theorem (*Lyons-Victoir Extension Theorem*). *For $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and any α-Holder function X, there is a function $\mathbb{X} : [0,T]^2 \to \mathbb{R}$ such that $(X, \mathbb{X})$ is an α-Holder rough path $\mathbb{X}$.*

Proof. In the full multidimensional setting, this is insanely non-trivial. Fortunately, it is easy in the single-dimensional case — we simply choose

$$\mathbb{X}_{s,t} = X_t \left(X_t - X_s \right).$$

□

It can actually be shown that this particular choice always gives a first-order calculus, like the Stratonovich stochastic calculus. Rough paths that exhibit a first-order calculus are called *geometric*, and are notable as they can be uniformly approximated by smooth paths (e.g. Wong-Zakai theorem).

Now for the key result:

Theorem. *For any α-Holder rough path with $\alpha > \frac{1}{3}$ and any differentiable F, the rough path integral exists:*

$$\int_0^T F(X)\, dX = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} \left[F(X_s)(X_t - X_s) + F'(X_s)\, \mathbb{X}_{s,t} \right].$$

If we want to integrate with respect to a more general function than just a function of X, we need to have a derivative term. The most general class we can integrate over are *controlled rough paths*: functions Y with some operator Y' such that

$$Y_{s,t} = Y'_s X_{s,t} + O\left(|t-s|^{2\alpha} \right).$$

Now, the rough path integral becomes

$$\int_0^T Y dX = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} \left[Y_s (X_t - X_s) + Y'_s \mathbb{X}_{s,t} \right],$$

which is now well-defined.

Let's think a little bit more about why this extra term is important. Perhaps the best way is to recall that the Ito and Stratanovich integrals are different: the $\mathbb{X}_{s,t}$ term is providing the extra information necessary to distinguish between these two integrals. For example, if we choose

$$\begin{aligned}\mathbb{W}_{s,t} &= \int_s^t W_{s,t} \circ dW_t \\ &= \left(\int_s^t W_t \circ dW_t\right) - W_s\left(W_t - W_s\right) \\ &= W_t\left(W_t - W_s\right),\end{aligned}$$

then the integral $\int F(W)\, dW$ corresponds exactly to the Stratanovich integral. Alternatively if

$$\begin{aligned}\mathbb{W}_{s,t} &= \int_s^t W_{s,t} dW_t \\ &= \left(\int_s^t W_t dW_t\right) - W_s\left(W_t - W_s\right) \\ &= W_t\left(W_t - W_s\right) - (t - s)\end{aligned}$$

then the integral $\int F(W)\, dW$ corresponds exactly to the Ito integral. For both of these, $(W, \mathbb{W})$ is a $\left(\frac{1}{2} - \epsilon\right)$-Holder rough path. What is cool is that $\mathbb{W}_{s,t}$ uniquely determines the integral.

In general, we have infinitely many different constructions of the rough path integral, depending on how we choose $\mathbb{X}_{s,t}$. In this way, the rough path integral can be seen as more general than the stochastic integral (semimartingales). However, since we are restricted to Holder-continuous processes, which are all continuous, it is also somewhat less general than the stochastic integral which allows for jumps (although, we can always deal with this by decomposing the integral into the continuous and discontinuous parts anyway). Personally, I like to think of the theory of rough paths and the usual stochastic calculus as different formulations which complement each other quite nicely. The former has nice ties to functional analysis, while the latter is more familiar in modern probability theory.

Assorted Results

- Ito's formula still holds, where $[X]_t$ is derived from an algebraic relation from X and $\mathbb{X}$, however, it requires that $f \in C_b^3$ not just $f \in C_b^2$.
- A priori estimates: For any solution Y_t to the *rough differential equation* (RDE):

$$Y_t = Y_0 + \int_0^t f\left(Y_t\right) dX_t,$$

  where $f \in C_b^2$ and $(X, \mathbb{X})$ is an α-Holder rough path, Y_t is α-Holder continuous, and a Holder constant (a choice of C) is *explicitly known.*

- Existence and uniqueness: The solution to the above RDE exists and is unique locally. We get this from standard contraction mapping arguments! Additionally, if $f \in C_b^3$ then the solution exists and is unique globally.

  The space of α-Holder functions forms a Banach space under a particular norm. By extension, the space of α-Holder rough paths also forms a complete metric space. This allows for many of the essential results in functional analysis to carry over into analysis of SDEs!

- Continuity for approximations: there is

$$\|Y - Y^\epsilon\|_\alpha \leq C_M \left(\left| Y_0 - Y_0^\epsilon \right| + \left\| X_{s,t} - X_{s,t}^\epsilon \right\|_\alpha + \left\| \mathbb{X}_{s,t} - \mathbb{X}_{s,t}^\epsilon \right\|_{2\alpha} \right).$$

  Results such as this immediately allow for one to explicitly provide convergence rates for numerical algorithms to solve RDEs (and SDEs too!). The Wong-Zakai theorem is a corollary of this.

- Integration with respect to non-semimartingales, e.g. fractional Brownian motion. You just need to be careful about the class of processes you are integrating over.
- Explicit solutions to linear SPDEs: since the theory is entirely deterministic, an extension of Feynman-Kac gives an explicit formula for solutions to linear SPDEs, which is perfect for Monte-Carlo estimation.
- Adaptation of traditional PDE theory to simple nonlinear SPDEs
- The theory of regularity structures is a (Fields medal winning) extension of the theory of rough paths into a powerful toolbox for analysing stochastic partial differential equations.
- For more results, check out the text "A Course on Rough Paths" by Friz and Hairer.